New approach for deriving operator identities by alternately using normally, antinormally, and Weyl ordered integration technique

Hong-Yi Fan and Hong-Chun Yuan
Department of Physics, Shanghai Jiao Tong University, Shanghai, 200030, P. R. China

Dirac's ket-bra formalism is the "language" of quantum mechanics and quantum field theory. In Ref.[1, 2] (Fan et al, Ann. Phys. 321 (2006) 480; 323 (2008) 500) we have reviewed how to apply Newton-Leibniz integration rules to Dirac's ket-bra projectors. In this work by alternately using the technique of integration within normal, antinormal, and Weyl ordering of operators we not only derive some new operator ordering identities, but also deduce some useful integration formulas regarding to Laguerre and Hermite polynomials. This opens a new route of deriving mathematical integration formulas by virtue of the quantum mechanical operator ordering technique.
Keywords: new approach for operator ordering identities; the IWOP technique

I. INTRODUCTION

In the foreword of the book ≪The Principles of Quantum Mechanics≫ Dirac wrote: "*The symbolic method, $\cdots$ enables one to express the physical law in a neat and concise way, and will probably be increasingly used in the future as it becomes better understood and its own special mathematics gets developed.*"[3] Following his expectation, the technique of integration within an ordered product (IWOP) of operators was invented which can directly apply Newton-Leibniz integration rule to ket-bra projective operators. [1, 2] The essence of IWOP technique is to convert these ket-bra operators into certain ordered product (normal ordering, antinormal ordering, and Weyl ordering) of bosonic creation and annihilation operators such that they can be treated as ordinary parameters while performing the integrations, but operators' essence will not lose in this approach. For example, using the IWOP technique[4, 5] we have converted the completeness relation of the coordinate eigenvector $|x\rangle$ as a pure Gaussian integration within the normal ordering : : ,

$$\int_{-\infty}^{\infty} dx\,|x\rangle\langle x| = \int_{-\infty}^{\infty}\frac{dx}{\sqrt{\pi}} :e^{-(x-X)^2}: \;= 1, \tag{1}$$

where $X=(a+a^\dagger)/\sqrt{2}$ is the coordinate operator with $\left[a,a^\dagger\right]=1$. It turns out that performing the ket-bra integration $\int_{-\infty}^{\infty}\frac{dx}{\mu}\left|\frac{x}{\mu}\right\rangle\langle x|$ leads to the single-mode squeezing operator. Recently, we directly and concisely obtain the connection between Wick ordered polynomial and Hermite polynomials[6] by virtue of the the IWOP technique.[7]

In this work we shall demonstrate how to derive some new operator ordering identities and new integration formulas regarding to Laguerre polynomials $L_n^{(\alpha)}(x)$ and Hermite polynomials $H_n(x)$ by alternately using the technique of integration within normal, antinormal, and Weyl ordering of operators. This opens a new route of deriving mathematical integration formulas by virtue of the quantum mechanical operator ordering technique.

II. ANTINORMALLY ORDERED EXPANSION OF THE OPERATOR X^n AND $H_n(X)$

To begin with, using the Baker-Hausdorff formula

$$e^{A+B}=e^Ae^Be^{-\frac{1}{2}[A,B]}=e^Be^Ae^{-\frac{1}{2}[B,A]} \tag{2}$$

and the generating function of $H_n(x)$

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x)=e^{2tx-t^2} \tag{3}$$

where

$$H_n(x)=\sum_{l=0}^{[n/2]}\frac{(-1)^l\,n!}{l!(n-2l)!}(2x)^{n-2l}, \tag{4}$$

we easily obtain

$$e^{\lambda X}=e^{\frac{\lambda}{\sqrt{2}}\left(a+a^{\dagger}\right)}=\vdots e^{\lambda X-\frac{1}{4}\lambda^{2}}\vdots=\sum_{n=0}^{\infty}\frac{\left(\frac{\lambda}{2}\right)^{n}}{n!}\vdots H_{n}\left(X\right)\vdots, \tag{5}$$

where $\vdots\,\vdots$ stands for antinormal ordering. Comparing Eq.(5) with the same power of λ in the expansion of $e^{\lambda X}=\sum_{n=0}^{\infty}\frac{\lambda^{n}}{n!}X^{n}$, we have the neat formula

$$X^{n}=2^{-n}\vdots H_{n}\left(X\right)\vdots, \tag{6}$$

which is just the antinormal ordering expansion of X^{n}. Taking $n=2$ for example,

$$\begin{aligned}\frac{1}{4}\vdots H_{2}\left(X\right)\vdots&=\frac{1}{4}\vdots\left(4X^{2}-2\right)\vdots\\&=\frac{1}{2}\left(a^{2}+2aa^{\dagger}+a^{\dagger 2}-1\right)=X^{2},\end{aligned} \tag{7}$$

as expected. It then follows from Eq.(6) that

$$X^{n+m}=2^{-n-m}\vdots H_{n+m}\left(X\right)\vdots=X^{n}X^{m}, \tag{8}$$

which implies the following new relation

$$\vdots H_{n+m}\left(X\right)\vdots=\vdots H_{n}\left(X\right)\vdots\vdots H_{m}\left(X\right)\vdots. \tag{9}$$

From the relation

$$\frac{d}{dx}H_{n}\left(x\right)=2nH_{n-1}\left(x\right), \tag{10}$$

and Eq.(6) we have

$$\frac{d}{dX}\vdots H_{n}\left(X\right)\vdots=2^{n}nX^{n-1}=2n\vdots H_{n-1}\left(X\right)\vdots=\vdots\frac{d}{dX}H_{n}\left(X\right)\vdots, \tag{11}$$

namely,

$$\frac{d}{dX}\vdots H_{n}\left(X\right)\vdots=\vdots\frac{d}{dX}H_{n}\left(X\right)\vdots. \tag{12}$$

This is another property of $\vdots H_{n}\left(X\right)\vdots$.

Moreover, from Eqs.(3) and (2) we see

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{t^{n}}{n!}H_{n}\left(\lambda X\right)&=e^{2t\lambda X-t^{2}}=\vdots e^{2t\lambda X-\left(\lambda^{2}+1\right)t^{2}}\vdots\\&=\sum_{n=0}^{\infty}\frac{\left(\sqrt{\left(\lambda^{2}+1\right)}t\right)^{n}}{n!}\vdots H_{n}\left(\frac{\lambda X}{\sqrt{\left(\lambda^{2}+1\right)}}\right)\vdots\end{aligned} \tag{13}$$

Comparing the same power of t on the two sides we obtain a new identity

$$H_{n}\left(\lambda X\right)=\left(\sqrt{\left(\lambda^{2}+1\right)}\right)^{n}\vdots H_{n}\left(\frac{\lambda X}{\sqrt{\left(\lambda^{2}+1\right)}}\right)\vdots \tag{14}$$

Especially, when $\lambda = 1$, Eq.(14) reduces to

$$H_n(X) = \left(\sqrt{2}\right)^n \vdots H_n\left(\frac{X}{\sqrt{2}}\right)\vdots. \tag{15}$$

which is different from Eq.(6). For $n = 2$ case,

$$\begin{aligned} 2\vdots H_2\left(\frac{X}{\sqrt{2}}\right)\vdots &= \vdots 2\left(a + a^\dagger\right)^2 - 4\vdots \\ &= 2\left(a^2 + 2aa^\dagger + a^{\dagger 2}\right) - 4 \\ &= 4\left(\frac{a + a^\dagger}{\sqrt{2}}\right)^2 - 2 \\ &= H_2(X) \end{aligned} \tag{16}$$

By analogy to the above derivation we can deduce the normal ordering expansion of X^n

$$X^n = (2i)^{-n} : H_n(iX) : , \tag{17}$$

and the normally ordered form of $H_n(X)$

$$H_n(X) = 2^n : X^n : . \tag{18}$$

III. NEW INTEGRATION FORMULAS ABOUT HERMITE POLYNOMIALS

Using the above results we can derive new integration formulas regarding to Hermite polynomials (HP) $H_n(x)$. Recalling the P-representation in the coherent state $|\beta\rangle$ basis[8]

$$\rho = \int \frac{d^2\beta}{\pi} P(\beta) |\beta\rangle\langle\beta|, \beta \equiv \beta_1 + i\beta_2 \tag{19}$$

where

$$|\beta\rangle = \exp\left[-\frac{|\beta|^2}{2} + \beta a^\dagger\right]|0\rangle, \tag{20}$$

is the coherent state satisfying with

$$a|\beta\rangle = \beta|\beta\rangle. \tag{21}$$

Utilizing Eqs.(6) and (19) as well as the vacuum projector $|0\rangle\langle 0| =: \exp\left(-a^\dagger a\right):$ and considering (17), we have

$$\begin{aligned} X^n &= 2^{-n} \int \frac{d^2\beta}{\pi} H_n\left(\frac{\beta + \beta^*}{\sqrt{2}}\right)|\beta\rangle\langle\beta| \\ &= 2^{-n}\int \frac{d^2\beta}{\pi} H_n\left(\sqrt{2}\beta_1\right) : \exp\left(-|\beta|^2 + \beta a^\dagger + \beta^* a - a^\dagger a\right) : \\ &= (2i)^{-n} : H_n(iX) : . \end{aligned} \tag{22}$$

Since within the normal ordering symbol a and $a^\dagger$ are commute, they can be treated as c-numbers. So we can set $a \to x$ and $a^\dagger \to y$, in this way we obtain

$$\int \frac{d^2\beta}{\pi} H_n\left(\sqrt{2}\beta_1\right)\exp\left(-|\beta|^2 + y\beta + x\beta^*\right) = i^{-n} H_n\left(i\frac{x + y}{\sqrt{2}}\right)e^{xy}, \tag{23}$$

which is a new integration formula. Here we derive it without really performing the integration in Eq.(23). This is an obvious advantage of the IWOP technique.

Next, the antinormally ordered expansion of an arbitrary bosonic operator in the coherent state $|\beta\rangle$ basis is[9]

$$\rho = \int \frac{d^2\beta}{\pi} \vdots \langle -\beta| \rho |\beta\rangle \exp[|\beta|^2 + \beta^* a - \beta a^\dagger + a^\dagger a] \vdots \tag{24}$$

provided that the integral is convergent, which shows that when ρ is in normal ordering, its coherent state matrix element $\langle -\beta| \rho |\beta\rangle$ can be immediately obtained. So we can get ρ's antinormally ordered expansion by just performing the integral of Eq.(24). According to Eqs.(17) and (24) and considering Eq.(6) again, we obtain the following equation

$$\begin{aligned} X^n &= (2i)^{-n} \int \frac{d^2\beta}{\pi} \vdots \langle -\beta| : H_n(iX) : |\beta\rangle \exp[|\beta|^2 + \beta^* a - \beta a^\dagger + a^\dagger a] \vdots \\ &= (2i)^{-n} \int \frac{d^2\beta}{\pi} \vdots H_n\left(-\sqrt{2}\beta_2\right) \exp[-|\beta|^2 + \beta^* a - \beta a^\dagger + a^\dagger a] \vdots \\ &= 2^{-n} \vdots H_n(X) \vdots . \end{aligned} \tag{25}$$

Thus, one immediately gets another new integration formula

$$\int \frac{d^2\beta}{\pi} H_n\left(-\sqrt{2}\beta_2\right) \exp[-|\beta|^2 + x\beta^* - \beta y] = i^n H_n\left(\frac{x+y}{\sqrt{2}}\right) e^{-xy}, \tag{26}$$

which is different from Eq.(23).

On the other hand, in Refs.[10, 11, 12], by considering P-representation of an operator ρ and using the Weyl ordering of the coherent state projector

$$|\beta\rangle\langle\beta| = 2 \,\vdots\, \exp\left[-2\left(\beta^* - a^\dagger\right)(\beta - a)\right] \,\vdots\,, \tag{27}$$

we have derived the useful formula which can convert ρ into its Weyl ordered form

$$\rho = 2 \int \frac{d^2\beta}{\pi} \,\vdots\, \langle -\beta| \rho |\beta\rangle \exp\left[2\left(\beta^* a - a^\dagger\beta + a^\dagger a\right)\right] \,\vdots\,, \tag{28}$$

where $\vdots\,\vdots$ stands for Weyl ordering, a and $a^\dagger$ are commute within $\vdots\,\vdots$. By using Eqs.(18) and (28), we obtain

$$\begin{aligned} H_n(X) &= 2^{n+1} \int \frac{d^2\beta}{\pi} \,\vdots\, \langle -\beta| : X^n : |\beta\rangle \exp\left[2\left(\beta^* a - a^\dagger\beta + a^\dagger a\right)\right] \,\vdots \\ &= i^n 2^{\frac{3n+2}{2}} \int \frac{d^2\beta}{\pi} \,\vdots\, \beta_2^n \exp\left[2\left(-|\beta|^2 + \beta^* a - a^\dagger\beta + a^\dagger a\right)\right] \,\vdots\,. \end{aligned} \tag{29}$$

Due to

$$H_n(X) = \,\vdots\, H_n(X) \,\vdots\,, \tag{30}$$

we have a new integration formula

$$i^n 2^{\frac{3n+2}{2}} \int \frac{d^2\beta}{\pi} \beta_2^n \exp\left[2\left(-|\beta|^2 + x\beta^* - y\beta\right)\right] = H_n\left(\frac{x+y}{\sqrt{2}}\right) e^{-2xy}, \tag{31}$$

without really performing this integration.

Finally, in the similar way for the momentum $P = \frac{1}{i\sqrt{2}}(a + a^\dagger)$, we easily obtain some integration formula as follows

$$\int \frac{d^2\beta}{\pi} H_n\left(\sqrt{2}\beta_2\right) \exp\left(-|\beta|^2 + y\beta + x\beta^*\right) = i^{-n} H_n\left(\frac{x-y}{\sqrt{2}}\right) e^{xy}, \tag{32}$$

$$\int \frac{d^2\beta}{\pi} H_n\left(\sqrt{2}\beta_1\right) \exp\left(-|\beta|^2 + x\beta^* - y\beta\right) = i^n H_n\left(\frac{x-y}{i\sqrt{2}}\right) e^{-xy}, \tag{33}$$

and

$$(-i)^n\, 2^{\frac{3n+2}{2}} \int \frac{d^2\beta}{\pi} \beta_1^n \exp\left[2\left(-|\beta|^2 + \beta^* a - a^\dagger\beta\right)\right] = H_n\left(\frac{x-y}{i\sqrt{2}}\right) e^{-2xy}. \tag{34}$$

IV. OPERATOR IDENTITIES ABOUT LAGUERRE POLYNOMIALS

Using Eq.(1) and the IWOP technique, we have the normally ordered expansion formula

$$\begin{aligned} e^{\lambda X^2} &= \int_{-\infty}^{\infty} dx e^{\lambda x^2} \left|x\right\rangle \left\langle x\right| =: \int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} e^{\lambda x^2} e^{-(x-X)^2} : \\ &= (1-\lambda)^{-1/2} : \exp[\frac{-\lambda X^2}{\lambda - 1}] : . \end{aligned} \tag{35}$$

Comparing Eq.(35) with the generating function of the Laguerre polynomials

$$(1-t)^{-\alpha-1} \exp[\frac{xt}{t-1}] = \sum_{n=0}^{\infty} L_n^{\alpha}(x) t^n, \tag{36}$$

we see when $\alpha = -1/2$,

$$: \sum_{n=0}^{\infty} L_n^{-1/2}\left(-X^2\right) \lambda^n : = e^{\lambda X^2}, \tag{37}$$

which is a new operator identity. Then comparing Eq.(37) with $e^{\lambda X^2} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} X^{2n}$, we see

$$X^{2n} = n! : L_n^{-1/2}\left(-X^2\right) : . \tag{38}$$

Due to $X^{2n} = (2i)^{-2n} : H_{2n}(iX) :$ in Eq.(17), we obtain the relation connecting Hermite polynomial and Laguerre polynomial

$$H_{2n}(iX) = (-1)^n 2^{2n} n! L_n^{-1/2}\left(-X^2\right), \tag{39}$$

this relation still holds when $iX \rightarrow x$, i.e.,

$$H_{2n}(x) = (-1)^n 2^{2n} n! L_n^{-1/2}\left(x^2\right), \tag{40}$$

which coincides with Ref.[13]. Since $H_{2n}(X) = 2^{2n} : X^{2n} :$ in Eq.(18), we also have

$$: X^{2n} : = (-1)^n n! L_n^{-1/2}\left(X^2\right). \tag{41}$$

Considering Eqs.(24) and (38), we derive

$$\begin{aligned} X^{2n} &= n! \int \frac{d^2\beta}{\pi} \vdots \left\langle -\beta\right| : L_n^{-1/2}\left(-X^2\right) : \left|\beta\right\rangle \exp[|\beta|^2 + \beta^* a - \beta a^\dagger + a^\dagger a] \vdots \\ &= n! \int \frac{d^2\beta}{\pi} \vdots L_n^{-1/2}\left(2\beta_2^2\right) \exp[-|\beta|^2 + \beta^* a - \beta a^\dagger + a^\dagger a] \vdots . \end{aligned} \tag{42}$$

It then follows from Eq.(6) that

$$n! \int \frac{d^2\beta}{\pi} L_n^{-1/2}\left(2\beta_2^2\right) \vdots \exp[-|\beta|^2 + \beta^* a - \beta a^\dagger + a^\dagger a] \vdots = 2^{-2n} \vdots H_{2n}(X) \vdots, \tag{43}$$

which is a new identity. Further, when $a \rightarrow x$ and $a^\dagger \rightarrow y$, we deduce a new integration formula as follows

$$n! \int \frac{d^2\beta}{\pi} L_n^{-1/2}\left(2\beta_2^2\right) \exp[-|\beta|^2 + x\beta^* - y\beta] = 2^{-2n} H_{2n}\left(\frac{x+y}{\sqrt{2}}\right) e^{-xy}. \tag{44}$$

In the similar manner, by considering Eqs.(28) and (41), we can obtain

$$\begin{aligned} (-1)^n n! L_n^{-1/2}\left(X^2\right) &= 2 \int \frac{d^2\beta}{\pi} \vdots \left\langle -\beta\right| : X^{2n} : \left|\beta\right\rangle \exp\left[2\left(\beta^* a - a^\dagger \beta + a^\dagger a\right)\right] \vdots \\ &= (-1)^n 2^{n+1} \int \frac{d^2\beta}{\pi} \vdots \beta_2^{2n} \exp\left[2\left(-|\beta|^2 + \beta^* a - a^\dagger \beta + a^\dagger a\right)\right] \vdots . \end{aligned} \tag{45}$$

From the above equation, due to $L_n^{-1/2}\left(X^2\right) = \vdots\, L_n^{-1/2}\left(X^2\right) \vdots$, we see that

$$2^{n+1}\int\frac{d^2\beta}{\pi}\beta_2^{2n}\vdots\,\exp\left[2\left(-\left|\beta\right|^2+\beta^* a-a^\dagger\beta+a^\dagger a\right)\right]\vdots = n!\vdots\, L_n^{-1/2}\left(X^2\right)\vdots\,. \tag{46}$$

It then follows the new integration formula

$$2^{n+1}\int\frac{d^2\beta}{\pi}\beta_2^{2n}\exp\left[2\left(-\left|\beta\right|^2+x\beta^*-y\beta\right)\right] = n!L_n^{-1/2}\left[\frac{1}{2}\left(x+y\right)^2\right]e^{-2xy}. \tag{47}$$

In summary, we have introduced an effective approach for deriving operator identities and new integration formulas by alternately using normally, antinormally, and Weyl ordered integration technique. This opens a new route of deriving mathematical integration formulas by virtue of the quantum mechanical operator ordering technique.

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